TEMPERATURE FIELD IN A PULSED DISCHARGE

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A theoretical description is given of the temperature field of a spark produced by a high-voltage pulsed discharge. The spark is considered to be an instantaneous volumetric heat source.

Activated particles which diffuse into the surrounding space are created in spark channels (temperature varies in the range 10^3-10^4°K) that are produced by a high-voltage pulsed discharge in a short period of time. Diffusion and formation of products occurs in a background of a spark temperature field that is dying out. Because of this, it is interest to consider the temperature field of individual sparks.

We represent the spark channel in the form of a finite cylinder of length 21 and diameter $2r_0$ heated uniformly over its entire volume to a temperatures T_0 at zero time. Essentially, this is equivalent to an instantaneous uniform release of heat in that same volume. In this case, the temperature in the spark channel at time t = 0 is

$$T_0 = q\frac{a}{\lambda}. \tag{1}$$

It is assumed the spark is struck in an isotropic uniform medium. The heat is propagated exclusively by thermal conductivity since transfer of heat by convection can be neglected because of the short lifetime of the spark (~ 10^{-7} sec). It is assumed the thermophysical coefficients of the medium are independent of temperature.

The equation for thermal conductivity in a medium moving with a velocity v has the form

$$T'_t = a\Delta T - vT'_x + \frac{a}{\lambda} f(x, y, z, t), \tag{2}$$

where f(x, y, z, t) is a function of the heat source.

1. We consider the case of the static mode (gas velocity v = 0). Transforming to cylindrical coordinates, we have in place of Eq. (2)

$$T'_t = a\left[T''_{rr} + \frac{1}{r}T'_r + T''_{zz}\right] + \frac{a}{\lambda} f(r, z, t), \tag{3}$$

where

$$f(r, z, t) = \begin{cases} q\delta(t) & |z| \leqslant l,\ r \leqslant r_0 \\ 0 & |z| > l,\ r > r_0. \end{cases} \tag{4}$$

Initial and boundary conditions for the problem are:

$$\begin{gathered} T(r, z, 0) = 0, \\ T'_r = T'_z|_{r,z=0} = 0 \text{ (because of symmetry)}, \\ T'_r = T'_z|_{r,z=\infty} = 0,\ T(r, z, t)|_{r,z=\infty} = 0. \end{gathered} \tag{5}$$

We shall seek a solution for the problem represented by Eqs. (3), (4), and (5) in two ways.

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A. We solve Eq. (3) for the initial and boundary conditions (4) and (5) indicated above by means of a Green's function

$$T(r,\ z,\ t) = \frac{a}{\lambda} \int_0^\infty dt' \int_0^\infty r'dr' \int_0^{2\pi} dz' G(r,\ z,\ \varphi,\ t|r',\ z',\ \varphi',\ t') f(r',\ z',\ \varphi',\ t'). \tag{6}$$

The function G is given in cylindrical coordinates by the expression

$$G = \frac{1}{[4\pi a(t-t')]^{3/2}} \exp\left[-\frac{r^2 + r'^2 - 2rr'\cos\varphi' + (z-z')^2}{4a(t-t')}\right]. \tag{7}$$

Substituting Eqs. (4) and (7) into Eq. (6) and integrating with respect to t' and allowing for the properties of the δ function, we have

$$T(r,\ z,\ t) = \frac{qa}{\lambda\,[4\pi at]^{3/2}} \int_{-l}^{l} dz' \int_0^{r_0} r'dr' \int_0^{2\pi} d\varphi' \exp\left[-\frac{r^2 + r'^2 - 2rr'\cos\varphi' + (z-z')^2}{4a(t-t')}\right]. \tag{8}$$

Then, carrying out the integration with respect to z and φ' [1], we find

$$T(r,\ z,\ t) = \left[\frac{q}{4\lambda t}\left[\operatorname{erf}\frac{l+z}{2\sqrt{at}} + \operatorname{erf}\frac{l-z}{2\sqrt{at}}\right] \int_0^{r_0} r'dr' \exp\left[-\frac{r^2 + r'^2}{4at}\right] I_0\left(\frac{rr'}{2at}\right). \tag{9}$$

The last integral can be expressed through the P-function tabulated in [2]. Equation (9) then transforms to

$$T(r,\ z,\ t) = \frac{qa}{2\lambda} P\left(\frac{r_0}{\sqrt{2at}},\ \frac{r}{\sqrt{2at}}\right)\left[\operatorname{erf}\frac{l+z}{2\sqrt{at}} + \operatorname{erf}\frac{l-z}{2\sqrt{at}}\right]. \tag{10}$$

Here

$$P\left(\frac{r_0}{\sqrt{2at}},\ \frac{r}{\sqrt{2at}}\right) = \frac{1}{2at} \int_0^{r_0} r'dr' \exp\left[-\frac{r^2 + r'^2}{4at}\right] I_0\left(\frac{rr'}{2at}\right). \tag{11}$$

Considering Eq. (1), we obtain

$$T(r,\ z,\ t) = \frac{1}{2} T_0 P\left(\frac{r_0}{\sqrt{2at}},\ \frac{r}{\sqrt{2at}}\right)\left[\operatorname{erf}\frac{l+z}{2\sqrt{at}} + \operatorname{erf}\frac{l-z}{2\sqrt{at}}\right]. \tag{12}$$

The resultant equations (10) and (12) also give a solution of the problem of the temperature field of a spark for a gas velocity v = 0.

Temperature on the channel axis, T(0, z, t), is determined from the following formula:

$$T(0,\ z,\ t) = \frac{1}{2} T_0 P\left(\frac{r_0}{\sqrt{2at}},\ \frac{0}{\sqrt{2at}}\right)\left[\operatorname{erf}\frac{l+z}{2\sqrt{at}} + \operatorname{erf}\frac{l-z}{2\sqrt{at}}\right]. \tag{13}$$

For r = 0, the P-function is

$$P\left(\frac{r_0}{\sqrt{2at}},\ \frac{0}{\sqrt{2at}}\right) = 1 - \exp\left(-\frac{r_0^2}{4at}\right);$$

therefore, we obtain for T(0, z, t) the analytic expression

$$T(0,\ z,\ t) = \frac{1}{2} T_0\left[1 - \exp\left(-\frac{r_0^2}{4at}\right)\right]\left[\operatorname{erf}\frac{l+z}{2\sqrt{at}} + \operatorname{erf}\frac{l-z}{2\sqrt{at}}\right]. \tag{14}$$

The temperature at the boundary of the channel, $T(r_0,\ z,\ t)$, is given by the equation

$$T(r_0,\ z,\ t) = \frac{1}{4} T_0\left[1 - \exp\left(-\frac{r_0^2}{2at}\right) I_0\left(\frac{r_0^2}{2at}\right)\right]\left[\operatorname{erf}\frac{l+z}{2\sqrt{at}} + \operatorname{erf}\frac{l-z}{2\sqrt{at}}\right]. \tag{15}$$

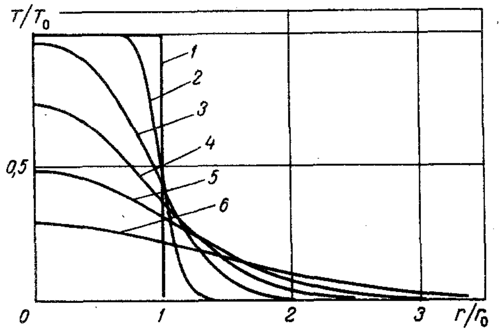

Fig. 1

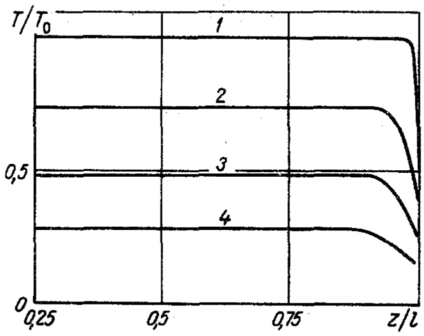

Fig. 2

Fig. 1. T/T_0 as a function of r/r_0: 1) at zero time; 2) after 10^{-6}; 3) 10^{-5}; 4) $2.5 \cdot 10^{-5}$; 5) $5 \cdot 10^{-5}$; 6) 10^{-4} sec.

Fig. 2. Temperature distribution in discharge gap: 1) after 10^{-6}; 2) $2.5 \cdot 10^{-5}$; 3) $5 \cdot 10^{-5}$; 4) $1 \cdot 10^{-4}$ sec.

B. Applying the Laplace–Hankel transformation

$$\psi(p, \sigma, z) = \int_0^\infty \exp(-pt)\, dt \int_0^\infty rT(r, z, t) J_0(\sigma r)\, dr$$

to Eq. (3), we obtain the operator equations

$$\begin{gathered} p\psi_1 = T_0 \int_0^{r_0} J_0(\sigma r)\, r dr + a(-\sigma^2 \psi_1 + \psi''_{1zz}), \\ p\psi_2 = a(-\sigma^2\psi_2 + \psi''_{2zz}). \end{gathered} \tag{16}$$

Here, the subscripts 1 and 2 refer respectively to the regions $|z| \le l$ and $|z| > l$. Furthermore, the boundary conditions transform to:

$$\begin{gathered} \psi'_{1z}\big|_{z=0} = 0, \quad \psi_2\big|_{z=\infty} = 0, \\ \psi_1\big|_{z=l} = \psi_2\big|_{z=l}, \quad \psi'_{1z}\big|_{z=l} = \psi'_{2z}\big|_{z=l}. \end{gathered} \tag{17}$$

The equation system (16) has the following solutions:

$$\begin{gathered} \psi_1 = A_1 \exp\left[-\sqrt{\frac{p}{a} + \sigma^2 z}\right] + A_2 \exp\left[\sqrt{\frac{p}{a} + \sigma^2 z}\right] + \frac{T_0 \int_0^{r_0} J_0(\sigma r)\, r dr}{p + a\sigma^2}, \\ \psi_2 = B_1 \exp\left[-\sqrt{\frac{p}{a} + \sigma^2 z}\right] + B_2 \exp\left[\sqrt{\frac{p}{a} + \sigma^2 z}\right]. \end{gathered}$$

The integral transforms found after substitution of the values of the coefficients A_1, A_2, B_1, and B_2 with the help of Eq. (17) have the form

$$\psi_1 = \frac{T_0 \int_0^{r_0} J_0(\sigma r') r' dr'}{p + a\sigma^2} \left\{1 - \frac{1}{2} \exp\left[-\sqrt{\frac{p}{a} + \sigma^2 (z+l)}\right] - \frac{1}{2} \exp\left[-\sqrt{\frac{p}{a} + \sigma^2 (l-z)}\right]\right\},$$

$$\psi_2 = \frac{T_0 \int_0^{r_0} J_0(\sigma r') r' dr'}{p + a\sigma^2} \left\{\exp\left[-\sqrt{\frac{p}{a} + \sigma^2 (z-l)} - \exp\left[-\sqrt{\frac{p}{a} + \sigma^2 (z+l)}\right]\right]\right\}.$$

The inverse transforms of the functions ψ_1 and ψ_2, in accordance with the expressions for the inverse Laplace–Hankel transformation, are given by a single expression

$$T(r, z, t) = \frac{1}{2} T_0 \left[\operatorname{erf} \frac{l+z}{2\sqrt{at}} + \operatorname{erf} \frac{l-z}{2\sqrt{at}}\right] \int_0^{r_0} r' dr' \int_0^\infty \exp(-a\sigma^2 t) J_0(\sigma r) J_0(\sigma r')\, \sigma d\sigma.$$

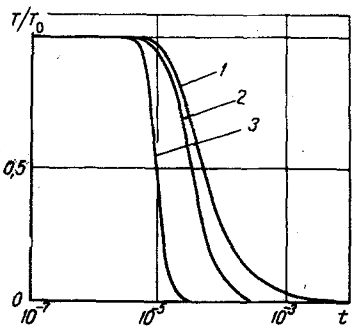

Fig. 3. Dependence of T/T_0 on t in the dynamic mode: 1) for v = 0; 2) 1.0 m/sec; 3) 5.0 m/sec.

Further, we obtain as the result of identity transformations [1]

$$T(r, z, t) = \frac{T_0}{4at}\left[\operatorname{erf}\frac{l+z}{2\sqrt{at}} + \operatorname{erf}\frac{l-z}{2\sqrt{at}}\right]\int_0^{r_0} r'dr' \exp\left[-\frac{r^2+r'^2}{4at}\right] I_0\left(\frac{rr'}{2at}\right).$$

Considering Eq. (11), we obtain for the temperature function

$$T(r, z, t) = \frac{1}{2} T_0 P\left(\frac{r_0}{\sqrt{2at}}, \frac{r}{\sqrt{2at}}\right)\left[\operatorname{erf}\frac{l+z}{2\sqrt{at}} + \operatorname{erf}\frac{l-z}{2\sqrt{at}}\right],$$

which agrees precisely with Eq. (12).

The calculated dependence of the reduced temperature T/T_0 on the dimensionless radius r/r_0 is shown in Fig. 1 for different times (at z = 0). In the calculations, we used the values: $a = 1.87 \cdot 10^{-5}$ m^2/sec for air [4], $2r_0 = 1.10^{-4}$ m, and $2l = 2 \cdot 10^{-3}$ m. It is clear from the figure that heat is transported an insignificant distance. Thus the region at a diameter $3r_0$ is heated only 10^{-4} sec after spark initiation, but the heating is negligible.

Equation (12), (14), and (15) indicate the form of temperature distribution along the discharge gap is independent of r. The temperature profile for the spark channel axis is shown in Fig. 2. The temperature T/T_0 remains practically constant over the entire length of the discharge gap, dropping sharply at the end to a value of 0.5 T/T_0 for z = 0. The value of the function T/T_0 is almost unchanged over the short times we are considering since the loss of heat along the x axis into the surrounding medium is negligibly small.

2. We consider the case of the dynamic mode $v \neq 0$. Using the substitution

$$T(x, y, z, t) = U(x, y, z, t)\exp\left[-\frac{v^2t}{4a} + \frac{vx}{2a}\right],$$

we bring Eq. (2) to the form

$$U_t' = a\Delta U + \frac{a}{\lambda} f(x, y, z, t)\exp\left[\frac{v^2t}{4a} - \frac{vx}{2a}\right].$$

Then transforming to cylindrical coordinates and applying the Green's function method, we obtain

$$T(r, z, \varphi, t) = \frac{qa}{8\pi t\lambda}\exp\left[-\frac{v^2t}{4a} + \frac{vr\cos\varphi}{2a}\right]\left[\operatorname{erf}\frac{l+z}{2\sqrt{at}} + \operatorname{erf}\frac{l-z}{2\sqrt{at}}\right]$$
$$\times\int_0^{r_0} r'dr' \exp\left[-\frac{r^2+r'^2}{4at}\right]\int_0^{2\pi}\exp\left[\frac{rr'}{2at}\cos(\varphi-\varphi') - \frac{vr'}{2a}\cos\varphi'\right]d\varphi'. \quad (18)$$

The integral with respect to φ' in Eq. (18) is not expressible in general form through known functions. We therefore confine ourselves to the two limiting cases: temperature distribution in the direction $\varphi = 0$ conciding with gas motion and in the opposite direction $\varphi = \pi$. In that case, integrating with respect to φ, we obtain

$$T(r,\ z,\ 0,\ t) = \frac{1}{2}\,T_0 P\left(\frac{r_0}{\sqrt{2at}},\ \frac{r - vt}{\sqrt{2at}}\right)\left[\operatorname{erf}\frac{l+z}{2\sqrt{at}} + \operatorname{erf}\frac{l-z}{2\sqrt{at}}\right], \tag{19}$$

$$T(r,\ z,\ \pi,\ t) = \frac{1}{2}\,T_0 P\left(\frac{r_0}{\sqrt{2at}},\ \frac{r + vt}{\sqrt{2at}}\right)\left[\operatorname{erf}\frac{l+z}{2\sqrt{at}} + \operatorname{erf}\frac{l-z}{2\sqrt{at}}\right]. \tag{20}$$

In the limiting case $v = 0$, Eqs. (19) and (20) transform identically into Eq. (12).

The time dependence of the reduced temperature on the spark channel axis is shown in Fig. 3 for various gas flow rates. Fall in temperature occurs very rapidly and T/T_0 (for $v = 1$ m/sec) practically goes to zero in a time $t = 5 \cdot 10^{-4}$ sec. A difference in the values of the function T/T_0 in the static and dynamic modes is only observed for $v > 0.5$ m/sec.

The equations (10), (12), (14), (15), (19), and (20) which were obtained make it possible to perform numerical calculations of the temperature fields in sparks both for a given value of T_0 and for known values of the energy and radius of a spark channel.

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